Quantum Information Processing (2024) 23:159
https://doi.org/10.1007/s11128-024-04367-0

Ergodicity and limit distribution of open quantum walks on the periodic graphs

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Received: 6 October 2023 / Accepted: 28 March 2024

Abstract
We discuss the limit distribution of open quantum walks on the periodic graphs, particularly on the cycles. We show that under certain hypothesis, we can benefit from the theory of the classical Markov chains. Thereby we can show that under certain condition the stationary distribution is unique. For certain models, we show directly the stationary distribution. We also notice that the open quantum walks cannot be always modeled as classical Markov chains by showing that it can break some classical probability rule. By providing with some examples, we show that there can be multiple stationary states for the open quantum walks on the cycles.

Keywords Open quantum walks · Classical Markov chain · Invariant distribution · Stationary states

Mathematics Subject Classification 60J10 · 81P47

1 Introduction

In this paper, we investigate the limit distribution of open quantum walks (OQWs hereafter) on the periodic graphs. We focus, however, on the cycles since the multi-dimensional extension can be done naturally.

Since the OQWs were introduced to model the quantum random walks, [1–3] their dynamical properties have been developed from the several viewpoints [4, 5, 8, 13, 14]. Recently, the present authors investigated the entropy production for the OQWs

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on the periodic graphs [9]. There one needs to consider a stationary state with which the entropy production is computed. It is a motivation to start the present work. On the other hand, it is also an interesting problem to consider the stationary states for the OQWs on the finite graphs comparing with the classical models. For a simple random walk on a finite connected graph $G = (V, E)$, it is well known that the stationary distribution $\pi = (\pi_i)$ is given by [6, 16] $\pi_i = \frac{d(i)}{2|E|}$, $i \in V$, where $d(i)$ is the degree of the vertex i. In particular, if G is d-regular, including the cycles, $\pi_i = \frac{1}{|V|}$, $i \in V$, i.e., π is the uniform distribution. Since the OQWs cannot be always modeled by classical random walks (Markov chains), the problem of finding the stationary states for OQWs is quite different from that for the classical random walks and it is not a simple question.

In this paper, in order to investigate the stationary distribution, we propose a hypothesis under which the problem of finding the stationary distributions of OQWs falls into that of classical random walks on an extended state space. Thereby it turns out to be an easy matter to show the stationary distribution for the OQWs once it satisfies the hypothesis. Also, for certain models, we directly compute the stationary distribution by using an elementary combinatorics.

The paper is organized as follows. In Sect. 2, we briefly introduce the definition of OQWs on the cycles and their dual processes. In Sect. 3, we introduce the hypothesis and investigate the stationary distributions of OQWs, the main result of this paper. In Sect. 4, we directly compute the limit distribution of OQW for certain models, namely when the generating matrices for the OQW commute with each other. In Sect. 5, we discuss the stationary states of OQWs. Particularly, we show that the stationary states may not be unique. In the Appendix, we characterize the cases for which the conditions of the hypothesis are fulfilled.

2 OQWs and their dual processes

In this section, we introduce the OQWs on the cycles and their dual processes, which appear when the Fourier transform is applied to the model. We start by introducing the OQWs.

2.1 OQW's on the cycles

Let us consider OQW's on a cycle $C_m = \{0, 1, \dots, m-1\}$. Let $\mathfrak{h} := l^2(C_m) \otimes \mathbb{C}^2 = l^2(C_m, \mathbb{C}^2) \cong \oplus_{i \in C_m} \mathbb{C}^2$ be the Hilbert space and $\mathfrak{A} := \oplus_{i \in C_m} \mathfrak{A}_i$ the von Neumann algebra, where $\mathfrak{A}_i \equiv \mathcal{M}_2 := \mathcal{B}(\mathbb{C}^2)$ is the algebra of 2×2 matrices for each $i \in C_m$. Notice that $\mathfrak{A}$ is a subalgebra of $\mathcal{B}(\mathfrak{h})$. Let B and C be 2×2 matrices such that

$$B^* B + C^* C = I_2, \tag{2.1}$$

the unit matrix. Consider the states (density matrices) on $\mathfrak{A}$ of the form:

$$\rho = \sum_{i \in C_m} \rho_i \otimes |i\rangle\langle i|, \tag{2.2}$$

where for each $i \in C_m$, ρ_i is a positive-definite operator (matrix) and satisfies $\sum_{i \in C_m} \mathrm{tr}(\rho_i) = 1$. Here $\{|i\rangle : i \in C_m\}$ denotes the canonical orthonormal basis of $l^2(C_m)$ and $|i\rangle\langle i|$ is an orthogonal projection onto the subspace generated by $|i\rangle$. Given a pair of matrices B and C as in (2.1), an OQW is an evolution of states of the form in (2.2) defined by

$$\mathcal{M}(\rho) = \sum_{i \in C_m} \rho_i' \otimes |i\rangle\langle i|, \quad \rho_i' = B\rho_{i+1}B^* + C\rho_{i-1}C^*. \tag{2.3}$$

Starting from an initial state $\rho^{(0)}$, the state at time $n \in \mathbb{N}$ is defined by

$$\rho^{(n)} = \mathcal{M}^n(\rho^{(0)}). \tag{2.4}$$

The random variable X_n is a position of the walker, a particle, at time n and it is distributed as

$$p^{(n)}(k) \equiv \mathbb{P}(X_n = k) := \mathrm{tr}(\rho_k^{(n)}), \quad k \in C_m. \tag{2.5}$$

Let $T : l^2(C_m) \to l^2(C_m)$ be the translation:

$$(Tf)(k) := f(k+1), \quad k \in C_m.$$

The adjoint of T is denoted by T^*. These operators can be naturally extended to $l^2(C_m, \mathbb{C}^2)$ as well as to $\oplus_{i \in C_m} \mathfrak{A}_i$ with the same symbols. Then the OQW evolution (2.4) can be rewritten as

$$\rho^{(n)} = \left(L_B R_{B^*} T + L_C R_{C^*} T^*\right)^n \rho^{(0)}, \tag{2.6}$$

here for $B \in \mathcal{B}(\mathbb{C}^2)$, L_B and R_B are the left and right multiplication operators, respectively, on $\mathcal{B}(\mathbb{C}^2)$:

$$L_B(A) = BA \text{ and } R_B(A) = AB, \quad A \in \mathcal{B}(\mathbb{C}^2),$$

and by abuse of notations we understand it also as operators on $\oplus_{i \in C_m} \mathfrak{A}_i$ by $L_B(\oplus_{i \in C_m} A_i) = \oplus_{i \in C_m} BA_i$ and $R_B(\oplus_{i \in C_m} A_i) = \oplus_{i \in C_m} A_i B$.

2.2 Dual process and the distribution

To compute the distribution of the OQW at time n, one has to compute the density operator from the evolution formula (2.5). However, as one notices, it is not simple to compute the density $\rho^{(n)}$ in the formula (2.6). The Fourier analysis may be helpful. In [10], the Fourier analysis was used to develop the dual process for the OQW.

Let C_m be a cycle of length m. Let $\widehat{C_m} := \{\chi_j : j = 0, 1, \dots, m-1\}$ be the space of characters of C_m, where χ_j, $j = 0, 1, \dots, m-1$, is defined by

$$\chi_j(k) = \omega^{jk}, \quad k \in C_m,$$

with ω the primitive mth root of unity: $\omega = e^{2\pi i/m}$. The Fourier transform of a function f on C_m is a function on $\widehat{C_m}$ defined by

$$\widehat{f}(\chi_j) = \sum_{k \in C_m} \chi_j(k) f(k).$$

The inverse transform is given by

$$f(k) = \frac{1}{m} \sum_{\chi \in \widehat{C_m}} \widehat{f}(\chi)\chi(-k).$$

The Fourier transform extends naturally to the operator valued functions. Therefore, the Fourier transform of the OQW evolution is given by

$$\begin{aligned} \widehat{\rho^{(n)}} &= \left(\widehat{\rho^{(n)}}(\chi)\right)_{\chi \in \widehat{C_m}}, \\ \widehat{\rho^{(n)}}(\chi_j) &= \left(\omega^{-j} L_B R_{B^*} + \omega^j L_C R_{C^*}\right)^n \widehat{\rho^{(0)}}(\chi_j), \quad j = 0, 1, \dots, m-1. \end{aligned}$$

The dual process is an evolution in the Fourier transform space defined by

$$Y_n(\chi_j) := \left(\omega^{-j} L_{B^*} R_B + \omega^j L_{C^*} R_C\right)^n (I_2). \tag{2.7}$$

Then, the distribution of the quantum walker is given by (see [10, Theorem 2.3] and its proof)

$$\begin{aligned} p^{(n)}(k) &= \frac{1}{m} \sum_{j=0}^{m-1} \omega^{-jk} \mathrm{tr}(\widehat{\rho^{(n)}}(\chi_j)) &(2.8) \\ &= \frac{1}{m} \sum_{j=0}^{m-1} \omega^{-jk} \mathrm{tr}(\widehat{\rho^{(0)}}(\chi_j) Y_n(\chi_j)). &(2.9) \end{aligned}$$

3 Ergodicity and limit distribution of OQWs on the cycles

In this section, we investigate the limit distributions of OQWs on the cycles. We have to compute the distributions given in the formula (2.9). Computation in the most general setting, as one guesses, is not so simple. Therefore, we are going to restrict ourselves to some classes of operators B and C. Before going into the details, we first observe a simple property in the OQWs, which may be well known. Let W be any 2×2 unitary matrix and define an automorphism $\tau = \tau_W$ on $\mathcal{M}_2$ by

$$\tau(A) := W^* A W, \quad A \in \mathcal{M}_2. \tag{3.1}$$

By abuse of notations the extension of τ to $\oplus_{i \in C_m} \mathfrak{A}_i$ is denoted by the same symbol. Our observation is that the OQW is invariant under the transformations of the generating matrices B and C by τ in (3.1) in the following sense.

Proposition 3.1 *The distributions of OQW generated by B and C with initial state $\rho^{(0)}$ are the same with the distributions of the OQW generated by $\tau(B)$ and $\tau(C)$ with initial state $\tau(\rho^{(0)})$.*

Proof Let $\rho^{(n)}$ be the state at time n of the OQW generated by B and C. For a notational convenience denote $A' = \tau(A)$. Notice that

$$\begin{aligned}
(\rho_i^{(n+1)})' &= W^*(B\rho_{i+1}^{(n)} B^*)W + W^*(C\rho_{i-1}^{(n)} C^*)W \\
&= B'(\rho_{i+1}^{(n)})' B'^* + C'(\rho_{i-1}^{(n)})' C'^*.
\end{aligned}$$

Therefore, $(\rho^{(n)})'$ is the state at time n of the OQW generated by B' and C' with initial state $(\rho^{(0)})'$. Since $\mathrm{tr}(\tau(A)) = \mathrm{tr}(A)$ for any $A \in \mathcal{M}_2$, the claim is verified. $\square$

3.1 Classical Markov chain on the pair of cycles

In this subsection, we introduce a classical Markov chain on an extended graph to facilitate the study of stationary distribution of the OQW. Let $G = \{(i, k) : i = 1, 2,\ k \in C_m\}$ be a pair of cycles. For the point (i, k), k stands for the site in C_m and i represents the chirality. Denoting $\{e_1, e_2\}$ the canonical basis of $\mathbb{C}^2$, let

$$P_1 := |e_1\rangle\langle e_1| = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \quad P_2 := |e_2\rangle\langle e_2| = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$$

be the orthogonal projections. On the graph G, define a transition matrix $P = (P_{(i,k),(j,l)})_{(i,k),(j,l) \in G}$ as follows:

$$P_{(i,k),(j,l)} = \begin{cases} \mathrm{tr}(P_i B^* P_j B), & \text{if } l = k-1, \\ \mathrm{tr}(P_i C^* P_j C), & \text{if } l = k+1, \\ 0, & \text{otherwise.} \end{cases} \tag{3.2}$$

It is easily shown that the matrix P defines a Markovian transition matrix.

Lemma 3.2 *The matrix P defined in (3.2) defines a transition matrix for a Markov chain on the graph G.*

Proof Fix a point $(i,k)\in G$. Then,

$$\begin{aligned}\sum_{(j,l)\in G} P_{(i,k),(j,l)} &= \sum_{j=1}^{2}\big(\mathrm{tr}(P_iB^*P_jB)+\mathrm{tr}(P_iC^*P_jC)\big)\\ &= \mathrm{tr}(P_i(B^*B+C^*C))\\ &= \mathrm{tr}(P_i)=1.\end{aligned}$$

This proves the claim. □

Given an initial distribution $\nu^{(0)}$ on G, the distribution $\nu^{(n)}$ of the Markov chain at time n, is given by

$$\nu^{(n)}=\nu^{(0)}P^n;\quad \nu^{(n)}(i,k)=\sum_{(j,l)}\nu^{(0)}(j,l)(P^n)_{(j,l),(i,k)},\ \ (i,k)\in G. \tag{3.3}$$

To analyze the asymptotic behavior in the general setting is far from the reach. So, our analysis will be done under some hypothesis. To introduce it, we start with an observation. Notice that from the relation $B^*B+C^*C=I_2$, the positive-definite matrices B^*B and C^*C commute with each other. Therefore, there are nonnegative numbers λ and μ satisfying $0\le\lambda\le 1$, $0\le\mu\le 1$ and a unitary matrix V such that

$$V^*B^*BV=\begin{pmatrix}\lambda & 0\\ 0 & \mu\end{pmatrix},\quad V^*C^*CV=\begin{pmatrix}1-\lambda & 0\\ 0 & 1-\mu\end{pmatrix}. \tag{3.4}$$

By Proposition 3.1, we can change B and C by $B'=\tau_V(B)$ and $C'=\tau_V(C)$, if necessary, and thereby we may assume V in (3.4) is the identity matrix I_2. Then by the singular value decomposition we may write

$$B=U_B\Sigma_B,\quad C=U_C\Sigma_C, \tag{3.5}$$

where U_B and U_C are 2×2 unitary matrices and

$$\Sigma_B=\begin{pmatrix}\sqrt{\lambda} & 0\\ 0 & \sqrt{\mu}\end{pmatrix},\quad \Sigma_C=\begin{pmatrix}\sqrt{1-\lambda} & 0\\ 0 & \sqrt{1-\mu}\end{pmatrix}.$$

Throughout this paper, we impose the following hypothesis.

(H) The maps $A\mapsto B^*AB$ and $A\mapsto C^*AC$ on $\mathcal{M}_2$ leave invariant the diagonal (commutative) subalgebra.

Remark 3.3 It turns out that the matrices that satisfy the hypothesis are one of the following forms.

$$\begin{pmatrix}a & 0\\ 0 & b\end{pmatrix},\ \begin{pmatrix}0 & a\\ b & 0\end{pmatrix},\ \begin{pmatrix}a & 0\\ b & 0\end{pmatrix},\ \begin{pmatrix}0 & a\\ 0 & b\end{pmatrix}.$$

For B and C of these type, the condition (3.5) holds automatically.

As the following proposition shows the hypothesis enables us to work with the classical theory of Markov chains. First let us define for $k \in C_m$, $i = 1, 2$, and $n \geq 0$,

$$p_i^{(n)}(k) := \mathrm{tr}(P_i \rho_k^{(n)}). \tag{3.6}$$

By (2.5), we notice that $\sum_{i=1}^2 p_i^{(n)}(k) = p^{(n)}(k)$, and hence we may say that $p_i^{(n)}(k)$ is the probability of finding the walker at time n at site k with its "chirality state, or spin state" i.

Proposition 3.4 *Assume the Hypothesis (H). Then, the distribution $p^{(n)}$ in (3.6) is equal to the distribution of the classical Markov chain at time n, namely, it holds that*

$$p_i^{(n)}(k) = \sum_{(j,l)\in G} p_j^{(0)}(l)(P^n)_{(j,l),(i,k)}, \quad (i,k) \in G. \tag{3.7}$$

Proof We have

$$\begin{aligned}
p_i^{(n+1)}(k) &= \mathrm{tr}(P_i \rho_k^{(n+1)}) \\
&= \mathrm{tr}(P_i (B \rho_{k+1}^{(n)} B^* + C \rho_{k-1}^{(n)} C^*)) \\
&= \mathrm{tr}(B^* P_i B \rho_{k+1}^{(n)} + C^* P_i C \rho_{k-1}^{(n)}) \\
&= \sum_{j=1}^2 \mathrm{tr}(P_j B^* P_i B \rho_{k+1}^{(n)} + P_j C^* P_i C \rho_{k-1}^{(n)}).
\end{aligned}$$

By the Hypothesis, we see that the matrices $B^* P_i B$ and $C^* P_i C$ are of diagonal form. Therefore, we have

$$P_j B^* P_i B = \mathrm{tr}(P_j B^* P_i B) P_j \text{ and } P_j C^* P_i C = \mathrm{tr}(P_j C^* P_i C) P_j.$$

Plugging into the last line, we get

$$\begin{aligned}
p_i^{(n+1)}(k) &= \sum_{j=1}^2 \left(p_j^n(k+1)\mathrm{tr}(P_j B^* P_i B) + p_j^{(n)}(k-1)\mathrm{tr}(P_j C^* P_i C) \right) \\
&= \sum_{(j,l)} p_j^{(n)}(l) P_{(j,l),(i,k)}.
\end{aligned}$$

Thus we obtained

$$p^{(n+1)} = p^{(n)} P.$$

Repeating the process, we conclude that

$$p^{(n)} = p^{(0)} P^n.$$

The proof is completed. □

Therefore, under the hypothesis (H), we can use the classical theory of Markov chains to investigate the distributions of the OQWs on the cycles. We will explore it in the next subsection.

3.2 Ergodicity and limit distributions

Once the hypothesis (H) is assumed, we have seen by Proposition 3.4 that the study of the distributions of the OQWs on the cycle reduces to the investigation of distributions of the classical Markov chains with transition matrices P given by (3.2). We need to classify the matrices B and C that satisfy (H). It is easy but tedious, and has a rather long exposition, so we leave it in the Appendix. The following is the main result of this paper.

Theorem 3.5 *Let X_n be the random variable of the position at time n of the OQW on the cycle C_m starting at the origin, which is generated by B and C. Suppose that the hypothesis (H) is satisfied and the cycle has a length of odd number. Then, the distribution of X_n converges to the uniform distribution on C_m.*

Proof By the hypothesis (H), the matrices $Y_n(\chi_l)$ in (2.7) are of diagonal forms for all $n \geq 0$. So, we can write

$$Y_n(\chi_l) = \begin{pmatrix} a_1^{(n)}(\chi_l) & 0 \\ 0 & a_2^{(n)}(\chi_l) \end{pmatrix}. \tag{3.8}$$

By the formula (2.7) we have the recursion relation:

$$\begin{aligned} Y_{n+1}(\chi_l) &= (\omega^{-l} L_{B^*} R_B + \omega^l L_{C^*} R_C) Y_n(\chi_l) \\ &= \sum_{i=1}^{2} a_i^{(n)}(\chi_l)(\omega^{-l} B^* P_i B + \omega^l C^* P_i C). \end{aligned} \tag{3.9}$$

We can now apply Lemma A.1. First, let us consider the Case 1 in the lemma. There are 4 further cases, but since all the cases can be dealt with similarly, we assume, for example, that

$$B^* P_1 B = \lambda P_1,\ \ B^* P_2 B = \mu P_2 \text{ and } C^* P_1 C = (1-\mu) P_2,\ \ C^* P_2 C = (1-\lambda) P_1.$$

Then from (3.9)

$$\begin{pmatrix} a_1^{(n+1)}(\chi_l) \\ a_2^{(n+1)}(\chi_l) \end{pmatrix} = \begin{pmatrix} \lambda\omega^{-l} a_1^{(n)}(\chi_l) + (1-\lambda)\omega^l a_2^{(n)}(\chi_l) \\ (1-\mu)\omega^l a_1^{(n)}(\chi_l) + \mu\omega^{-l} a_2^{(n)}(\chi_l) \end{pmatrix}. \tag{3.10}$$

Let us define for $k \in C_m$, $i = 1, 2$, and $n \geq 0$,

$$q_i^{(n)}(k) := \frac{1}{m} \sum_{l=0}^{m-1} \omega^{-lk} a_i^{(n)}(\chi_l). \tag{3.11}$$

By considering the initial state $\rho^{(0)} = \begin{pmatrix} \alpha & 0 \\ 0 & \beta \end{pmatrix}$, $\alpha + \beta = 1$, we have the relation:

$$\begin{aligned} p^{(n)}(k) &= \frac{1}{m} \sum_{j=0}^{m-1} \omega^{-jk} \mathrm{tr}(\widehat{\rho^{(0)}}(\chi_j) Y_n(\chi_j)) \\ &= \frac{1}{m} \sum_{j=0}^{m-1} \omega^{-jk} \mathrm{tr}((\alpha P_1 + \beta P_2) Y_n(\chi_j)) \\ &= \frac{1}{m} \sum_{j=0}^{m-1} \omega^{-jk} (\alpha a_1^{(n)}(\chi_j) + \beta a_2^{(n)}(\chi_j)) \\ &= \alpha q_1^{(n)}(k) + \beta q_2^{(n)}(k). \end{aligned} \tag{3.12}$$

In particular, taking the pairs $(\alpha, \beta) = (1, 0)$ and $(\alpha, \beta) = (0, 1)$, we get the normalization:

$$\sum_{k \in C_m} q_i^{(n)}(k) = 1, \quad i = 1, 2. \tag{3.13}$$

We multiply ω^{-lk} to both sides of (3.10), sum over l, and then divide by m to get

$$\begin{pmatrix} q_1^{(n+1)}(k) \\ q_2^{(n+1)}(k) \end{pmatrix} = \begin{pmatrix} \lambda q_1^{(n)}(k+1) + (1-\lambda) q_2^{(n)}(k-1) \\ (1-\mu) q_1^{(n)}(k-1) + \mu q_2^{(n)}(k+1) \end{pmatrix}. \tag{3.14}$$

Now let us consider a Markov chain on the graph G of a pair of cycles: $\{(i, k) : i = 1, 2,\ k \in C_m\}$, see the left in Fig. 1. The transition matrix $P = (P_{(i,k),(j,l)})$ for the Markov chain is defined by

$$\begin{aligned} &P_{(1,k),(1,k+1)} = \lambda,\ \ P_{(1,k),(2,k-1)} = 1 - \lambda, \\ &P_{(2,k),(2,k+1)} = \mu,\ \ P_{(2,k),(1,k-1)} = 1 - \mu,\ \ k \in C_m. \end{aligned}$$

Then, Eq. (3.14) is nothing but the Chapman–Kolmogorov equation for the Markov chain, namely, the function $q_i^{(n)}(k)$ is given by

$$q_i^{(n)}(k) = \sum_{(j,l)} P^n_{(i,k),(j,l)} q_j^{(0)}(l), \quad (i, k) \in \{1, 2\} \times C_m. \tag{3.15}$$

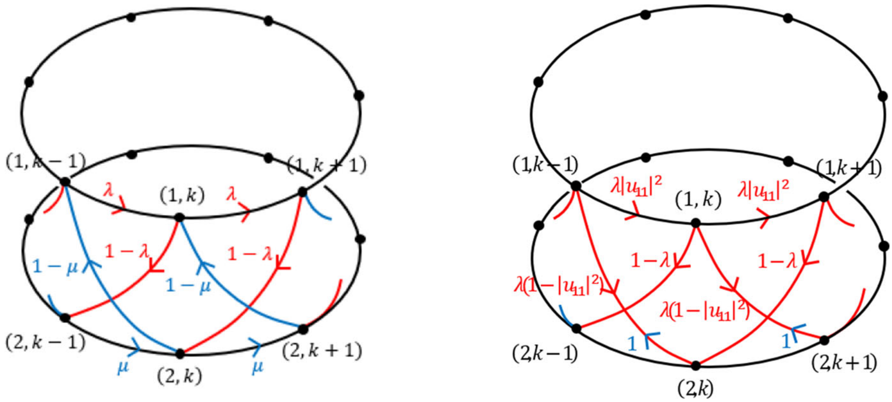

Fig. 1 Transition rules on the pair of cycles: Case 1 (left) and Case 2 (right) in Lemma A.1

As one can easily see from Fig. 1 the Markov chain is irreducible and ergodic, namely, it is irreducible, positive recurrent, and aperiodic. Therefore, there exists a unique invariant state $\pi = (\pi_{(i,k)})_{(i,k)}$ and it holds that (see [7, pp. 227, 243])

$$\lim_{n\to\infty} P^n_{(i,k),(j,l)} = \pi_{(j,l)}, \quad \text{for all } (i,k),\ (j,l) \in \{1,2\} \times C_m. \tag{3.16}$$

By taking the limit $n \to \infty$ in (3.15) we get

$$\lim_{n\to\infty} q_i^{(n)}(k) = \sum_{(j,l)} \pi_{(j,l)} q_j^{(0)}(l). \tag{3.17}$$

Since the right hand side of (3.17) is independent of (i,k), it means that the limit in the left hand side is a constant. By the normalization condition (3.13), we have $\sum_{(i,k)\in\{1,2\}\times C_m} q_i^{(n)}(k) = 2$ for all $n \geq 0$, and hence the constant should be $1/m$:

$$\lim_{n\to\infty} q_i^{(n)}(k) = \frac{1}{m}, \quad i = 1, 2. \tag{3.18}$$

Now for any initial state $\rho^{(0)} = \begin{pmatrix} \alpha & 0 \\ 0 & \beta \end{pmatrix}$, $\alpha + \beta = 1$, from (3.12) and (3.18) we see that

$$\lim_{n\to\infty} p^{(n)}(k) = \lim_{n\to\infty} \left(\alpha q_1^{(n)}(k) + \beta q_2^{(n)}(k)\right) = \frac{1}{m}.$$

Next, we consider the Case 2 in Lemma A.1. There are two possibilities (see the Appendix) and we suppose one of them (the other case is similar):

$$B^* P_1 B = \lambda |u_{11}|^2 P_1,\ \ B^* P_2 B = \lambda(1 - |u_{11}|^2) P_1, \quad C^* P_1 C = P_2,\ \ C^* P_2 C = (1-\lambda) P_1.$$

Then, following the preceding steps we arrive at the relation

$$\begin{pmatrix} q_1^{(n+1)}(k) \\ q_2^{(n+1)}(k) \end{pmatrix} = \begin{pmatrix} \lambda|u_{11}|^2 q_1^{(n)}(k+1) + \lambda(1-|u_{11}|^2) q_2^{(n)}(k+1) + (1-\lambda) q_2^{(n)}(k-1) \\ q_1^{(n)}(k-1) \end{pmatrix}. \tag{3.19}$$

We consider a Markov chain on the graph G of a pair of cycles again. This time the transition rules are depicted in the right diagram of Fig. 1. In this case one easily deduce that the Markov chain is irreducible and ergodic. Now, by the same argument used for the Case 1, we show the claim. The other cases are similar. This completes the proof. $\square$

Remark 3.6 (i) In Theorem 3.5, an odd number in the parity of the length of the cycle is indispensable. Let us consider the example: $B = C = \frac{1}{\sqrt{2}}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ on C_4. The condition (H) holds, but there is no limit distribution. In fact, let us define

$$\rho_0 := \begin{pmatrix} \alpha & 0 \\ 0 & \beta \end{pmatrix}, \quad \overline{\rho}_0 := \begin{pmatrix} \beta & 0 \\ 0 & \alpha \end{pmatrix}, \quad \alpha + \beta = 1.$$

Starting from $\rho^{(0)} = \rho_0 \otimes |0\rangle\langle 0|$, we have

$$\begin{aligned} \rho^{(2n-1)} &= \frac{1}{2}\overline{\rho}_0 \otimes |1\rangle\langle 1| + \frac{1}{2}\overline{\rho}_0 \otimes |3\rangle\langle 3|, \\ \rho^{(2n)} &= \frac{1}{2}\rho_0 \otimes |0\rangle\langle 0| + \frac{1}{2}\rho_0 \otimes |2\rangle\langle 2|, \quad n \geq 1. \end{aligned}$$

(ii) We have shown that under the hypothesis (H), the limit distribution of an OQW is uniform. However, it is worth noticing that it does not mean that the stationary distribution $\pi = (\pi_{(i,k)})_{(i,k)}$ in (3.16) is uniform in the chirality state space. For example, let us consider the case 1 (see the left diagram of Fig. 1) on C_3 with $\lambda = 1/3$, $\mu = 1/4$. One can show that the stationary distribution is given by

$$\pi_{(i,k)} = \begin{cases} 3/17, & i = 1, \ k = 0, 1, 2, \\ 8/51, & i = 2, \ k = 0, 1, 2. \end{cases}$$

4 Direct computation of the limit distribution for certain models

In this section we directly compute the distribution for certain models. The main method is to use the Fourier analysis through the formula (2.9). It is still hard to tackle the problem in the most general setting. Nevertheless, we can deal with quite interesting models. Here we consider the commuting cases.

Suppose that $[B^\sharp, C] = 0$, where $B^\sharp = B$ or B^*. For example, let $p, q \in [0, 1]$ be such that $p + q = 1$ and let $B = \begin{pmatrix} 1 & 0 \\ 0 & \sqrt{p} \end{pmatrix}$ and $C = \begin{pmatrix} 0 & 0 \\ 0 & \sqrt{q} \end{pmatrix}$. This pair, of course, satisfy the hypothesis (H). However, there are examples that do not satisfy (H). For example, let $B = \sqrt{p}U$ and $C = \sqrt{q}V$ with commuting unitaries U and V. Anyway, when B and C commute, we have the following result.

Theorem 4.1 *Let X_n be the random variable of the position at time n of the OQW on the cycle C_m of length m, which is generated by B and C and starts at the origin. Suppose that B and B^* commute with C. Then the distribution of X_n converges to the uniform distribution on C_m as n goes to infinity.*

Proof Let

$$\rho^{(0)} = \oplus_{k\in C_m}\rho_k^{(0)}; \quad \rho_k^{(0)} = \begin{cases} \rho_0, & k = 0, \\ 0, & \text{otherwise.} \end{cases}$$

First, under the hypothesis of the theorem we see that

$$\begin{aligned} Y_n(\chi_j) &= \left(\omega^{-j} L_{B^*} R_B + \omega^j L_{C^*} R_C\right)^n (I_2) \\ &= \sum_{l=0}^{n} \binom{n}{l} \omega^{-jl} (B^*B)^l \omega^{j(n-l)} (C^*C)^{n-l}. \end{aligned}$$

Therefore by (2.9),

$$\begin{aligned} p^{(n)}(k) &= \sum_{l=0}^{n} \binom{n}{l} \frac{1}{m} \sum_{j=0}^{m-1} \omega^{j(n-2l-k)} \mathrm{tr}(\rho_0 (B^*B)^l (C^*C)^{n-l}) \\ &= \sum_{l=0}^{n} \binom{n}{l} \delta_{k,(n-2l)(\mathrm{mod}\ m)} \mathrm{tr}(\rho_0 (B^*B)^l (C^*C)^{n-l}). \end{aligned} \tag{4.1}$$

By assumption, B and B^* commute with C and so we can simultaneously diagonalize B^*B and C^*C. Using the property $B^*B + C^*C = I_2$, we may assume that

$$B^*B = \begin{pmatrix} \lambda & 0 \\ 0 & \mu \end{pmatrix}, \quad C^*C = \begin{pmatrix} 1-\lambda & 0 \\ 0 & 1-\mu \end{pmatrix}.$$

In this representation, suppose that

$$\rho_0 = \begin{pmatrix} a & b \\ c & d \end{pmatrix}, \quad a, d \geq 0 \text{ and } a + d = 1.$$

Plugging into (4.1), we get

$$p^{(n)}(k) = \sum_{l=0}^{n} \binom{n}{l} \delta_{k,(n-2l)(\mathrm{mod}\ m)} \left(a\lambda^l (1-\lambda)^{n-l} + d\mu^l (1-\mu)^{n-l}\right).$$

Now the proof of the theorem is completed by the following Proposition 4.2. □

Proposition 4.2 *Let $p, q > 0$ be such that $p + q = 1$. Then, for any $k \in C_m = \{0, 1, \cdots, m-1\}$,*

$$\lim_{n\to\infty} \sum_{l=0}^{n} \binom{n}{l} \delta_{k,(n-2l)(\bmod m)} p^l q^{n-l} = \frac{1}{m}.$$

For the proof of the proposition, we list some simple observations. For any real number a let $\lfloor a \rfloor$ denote the integer part of a.

Lemma 4.3 *Under the same condition as in Proposition 4.2, let $l_0 = l_0(n, p, q) := \lfloor (n+1)p \rfloor$. Then the following properties hold:*

(i) *Given $n \in \mathbb{N}$, the sequence $\{1, 2, \ldots, n\} \ni l \mapsto \binom{n}{l} p^l q^{n-l}$ increases up to $l = l_0$ and then it decreases after l_0,*
(ii) $\binom{n}{l_0} p^{l_0} q^{n-l_0} = \mathrm{O}(\frac{1}{\sqrt{n}})$,
(iii) $\lim_{n\to\infty} \left(\sum_{l=0}^{m} \binom{n}{l} p^{n-l} q^l + \sum_{l=n-m}^{n} \binom{n}{l} p^{n-l} q^l\right) = 0.$

Proof Directly computing,

$$\frac{\binom{n}{l} p^l q^{n-l}}{\binom{n}{l-1} p^{l-1} q^{n-(l-1)}} \geq 1 \Longleftrightarrow (n+1)p \geq l,$$

and (i) follows. For (ii), use Stirling's formula and $l_0 \sim np$, to have

$$\binom{n}{l_0} p^{l_0} q^{n-l_0} \sim \frac{\sqrt{2\pi}\, n^{n+\frac{1}{2}} e^{-n}}{2\pi (np)^{np+\frac{1}{2}} (qn)^{nq+\frac{1}{2}} e^{-n}} p^{np} q^{nq} \sim 1/\sqrt{n}.$$

Finally, (iii) follows from (i) and (ii). □

Proof of Proposition 4.2 Let $\underline{n} := -n \ (\bmod m)$. By changing l to $n-l$ in the summation, we have

$$a(n,k) := \sum_{l=0}^{n} \binom{n}{l} \delta_{k,(n-2l)(\bmod m)} p^l q^{n-l} = \sum_{l=0}^{n} \binom{n}{l} \delta_{k',2l(\bmod m)} q^l p^{n-l}, \quad (4.2)$$

where

$$k' = (k - \underline{n}) \, (\bmod m). \quad (4.3)$$

According to the parity of k' we have

$$a(n,k) = \sum_{u=0}^{b} \binom{n}{um + k'/2} q^{um+k'/2} p^{n-(um+k'/2)}, \text{ if } k' \text{ is even}, \quad (4.4)$$

where $b = \max\{u \in \mathbb{N} : um + k'/2 \le n\}$, and

$$a(n,k) = \sum_{u=0}^{b} \binom{n}{((2u+1)m+k')/2} q^{((2u+1)m+k')/2} p^{n-((2u+1)m+k')/2}, \text{ if } k' \text{ is odd,} \tag{4.5}$$

where $b = \max\{u \in \mathbb{N} : ((2u+1)m+k')/2 \le n\}$. By Lemma 4.3, we can check that given any $\epsilon > 0$ if n is sufficiently large, for any $k_1, k_2 \in C_m$,

$$a(n,k_2) - \epsilon \le a(n,k_1) \le a(n,k_2) + \epsilon. \tag{4.6}$$

For example, let us compare $a(n,k_1)$ and $a(n,k_2)$ for which k_1' and k_2' are even, where k_1', k_2' follow the definition (4.3). Without loss of generality we may assume $k_1' \le k_2'$. Let $u_0 := \max\{u : um + k_2'/2 \le l_0,\ 0 \le u \le b\}$. Then, the partial sum of $a(n,k_2)$ up to $u = u_0$ is greater than that of $a(n,k_1)$ by Lemma 4.3 (i). Next we kick out the (u_0+1)th term from $a(n,k_1)$ while the corresponding term from $a(n,k_2)$ remains there, we compare term by term the remaining terms from $a(n,k_1)$ and $a(n,k_2)$. Again by Lemma 4.3 (i), the second half of $a(n,k_2)$ is greater than that of $a(n,k_1)$ with one term missing in the beginning. The missing term is of order $1/\sqrt{n}$ by Lemma 4.3 (ii) and a possible extra term is also sufficiently small by Lemma 4.3 (iii). Therefore, given $\epsilon > 0$, if n is sufficiently large, we get $a(n,k_1) \le a(n,k_2) + \epsilon$. To obtain the other inequality, $a(n,k_2) \le a(n,k_1) + \epsilon$, we just count from the second term in the summation of $a(n,k_1)$. Then first partial sum of $a(n,k_1)$ is now greater than that of $a(n,k_2)$ and with similar trick as above, we reach at the goal.

Now we obtained with n sufficiently large, for any $k \in C_m$,

$$m(a(n,k) - \epsilon) \le \sum_{k=0}^{m-1} a(n,k) \le m(a(n,k) + \epsilon).$$

But, obviously we have $\sum_{k=0}^{m-1} a(n,k) = 1$ and the proof is completed. □

5 Stationary states

In this section, we discuss the stationary states for the OQWs on the cycles. In the previous section, we have used the classical Markov chain models for the distribution of the OQWs. We first, however, notice the limitation of the classical models for the quantum dynamical systems.

5.1 Limitation of classical model for OQWs

Until now we benefited from the theory of classical Markov chains for the computation of limit distributions of OQWs. As one guesses, however, this idea can not be applied to every model. In this subsection we discuss such a limitation. Let us begin by giving an example showing that without the hypothesis the relation (3.7) does not hold in general. Let us consider an example dealt with in [10, Example 5]:

$$B = \frac{1}{\sqrt{3}}\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \quad C = \frac{1}{\sqrt{3}}\begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix}. \tag{5.1}$$

This example does not fulfill the hypothesis because, e.g., $B^* P_1 B$ is not a diagonal matrix. Let us consider an initial distribution $\nu^{(0)}$ such that $\nu^{(0)}(i,k) = \delta_{(i,k),(1,0)}$. Then the distribution $\nu^{(2)}$ at time 2 under the Markov chain with transition matrix P in (3.2) is directly computed as

$$\nu^{(2)}(1,-2) = 1/9, \ \nu^{(2)}(1,0) = 3/9,$$
$$\nu^{(2)}(2,0) = 2/9, \ \nu^{(2)}(1,2) = 1/9, \ \nu^{(2)}(2,2) = 2/9.$$

On the other hand, the distribution of the quantum walker defined in (3.6) at time 2 with initial state $\rho^{(0)} = P_1 \otimes |0\rangle\langle 0|$ results in

$$p_1^{(2)}(-2) = 1/9, \ p_1^{(2)}(0) = 1/9, \ p_2^{(2)}(0) = 2/9, \ p_1^{(2)}(2) = 1/9, \ p_2^{(2)}(2) = 4/9.$$

Thus the two distributions $p^{(2)}$ and $\nu^{(2)}$ are not equal to each other even though $p^{(0)} = \nu^{(0)}$.

Even further, this example shows that every OQWs cannot be classically modeled. It can be realized by showing that in quantum channels it may occur some classical violation. Let us consider any three ± 1-valued random variables ξ_1, ξ_2, ξ_3 on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$. The following inequality holds trivially:

$$\mathbb{P}(\xi_1 = 1, \xi_3 = 1) \le \mathbb{P}(\xi_1 = 1, \xi_2 = -1) + \mathbb{P}(\xi_2 = 1, \xi_3 = 1). \tag{5.2}$$

Recalling the projections $P_\alpha, \alpha = 1, 2$, introduced in the Sect. 3.1, we define for each state $\rho = \sum_{i \in C_m} \rho_i \otimes |i\rangle\langle i|$

$$S_{i,\alpha}(\rho) := P_\alpha \rho_i P_\alpha, \tag{5.3}$$

the intensity of the spin at site i to be up ($\alpha = 1$) or down ($\alpha = 2$). Therefore, in the state $\rho = \sum_{i \in C_m} \rho_i \otimes |i\rangle\langle i|$, the probability of the spin at site i to be up under the condition that the particle is found at site i is

$$p_{i,1}(\rho) = \frac{\mathrm{tr}(P_1 \rho_i)}{\mathrm{tr}(\rho_i)}. \tag{5.4}$$

Now let us consider the OQW on the cycle C_m with generating matrices B and C given in (5.1). We prepare the initial condition $\rho^{(0)} = \frac{1}{2} I_2 \otimes |0\rangle\langle 0|$. We define the following three ± 1-valued random variables ξ_1, ξ_2, and ξ_3 as follows.

$$\xi_1 = 1\ (-1) \text{ if the spin at the origin at time 0 is up (down)}$$

$$\xi_2 = 1\ (-1) \text{ if the spin at site } -1 \text{ at time } 1 \text{ is up (down)}$$
$$\xi_3 = 1\ (-1) \text{ if the spin at site } 2 \text{ at time } 4 \text{ is up (down).}$$

We claim that the inequality (5.2) fails. In fact, we compute

$$\mathbb{P}(\xi_1 = 1, \xi_3 = 1) = \mathbb{P}(\xi_1 = 1)\mathbb{P}(\xi_3 = 1|\xi_1 = 1).$$

We see that

$$\mathbb{P}(\xi_1 = 1) = \mathrm{tr}(P_1(\frac{1}{2}I_2)) = \frac{1}{2}.$$

Now to compute $\mathbb{P}(\xi_3 = 1|\xi_1 = 1)$, we notice that the condition $\xi_1 = 1$ means that a measurement of the spin at the origin was carried out at time 0 and the result was up. Then, the state was refreshed to be $\rho^{(0)} = P_1 \otimes |0\rangle\langle 0|$. Now to carry out another measurement at time 4 we need to prepare the state $\rho^{(4)} = \mathcal{M}^4(\rho^{(0)})$. Then, we measure the spin at site 2. Hence the probability of the event $\xi_3 = 1$ is

$$\begin{aligned}&\mathrm{tr}(P_1\{BC^3P_1(BC^3)^* + CBC^2P_1(CBC^2)^*\\&+C^2BCP_1(C^2BC)^* + C^3BP_1(C^3B)^*\}) = \frac{2}{27}.\end{aligned}$$

Therefore, we computed

$$\mathbb{P}(\xi_1 = 1, \xi_3 = 1) = \frac{1}{2} \cdot \frac{2}{27} = \frac{1}{27}.$$

We continue the similar computations to get

$$\begin{aligned}\mathbb{P}(\xi_1 = 1, \xi_2 = -1) &= \mathbb{P}(\xi_1 = 1)\mathbb{P}(\xi_2 = -1|\xi_1 = 1)\\&= \frac{1}{2}\mathrm{tr}(P_2BP_1B^*) = \frac{1}{2} \cdot 0 = 0.\\\mathbb{P}(\xi_2 = 1, \xi_3 = 1) &= \mathbb{P}(\xi_2 = 1)\mathbb{P}(\xi_3 = 1|\xi_2 = 1)\\&= \mathrm{tr}(P_1\{B(\frac{1}{2}I_2)B^*\}) \cdot \mathrm{tr}(P_1C^3P_1(C^3)^*) = \frac{1}{3} \cdot \frac{1}{27}.\end{aligned}$$

We see that the inequality (5.2) fails.

5.2 Non-uniqueness of stationary states

We consider a related quantum dynamical system on $\mathcal{B}(\mathfrak{h})$ defined by some Kraus operators. For each $i \in C_m$ define linear operators $L_{i,\pm}$ in $\mathcal{B}(\mathfrak{h})$ as follows.

$$\begin{aligned}L_{i,+} &:= B \otimes |i+1\rangle\langle i|,\\L_{i,-} &:= C \otimes |i-1\rangle\langle i|.\end{aligned}$$

We notice that

$$\sum_{i\in C_m}(L_{i,+}^*L_{i,+}+L_{i,-}^*L_{i,-})=I_{\mathfrak{h}}. \tag{5.5}$$

Let $\mathcal{L}$ be a Lindblad generator for a quantum Markov semigroup on $\mathcal{B}(\mathfrak{h})$ defined by [15]

$$\begin{aligned}\mathcal{L}(a) :=& -\frac{1}{2}\sum_{i\in C_m}\Big((L_{i,+}^*L_{i,+}+L_{i,-}^*L_{i,-})a-2(L_{i,+}^*aL_{i,+}\\ &+L_{i,-}^*aL_{i,-})+a(L_{i,+}^*L_{i,+}+L_{i,-}^*L_{i,-})\Big)\\ =& \sum_{i\in C_m}(L_{i,+}^*aL_{i,+}+L_{i,-}^*aL_{i,-})-a, \quad a\in\mathcal{B}(\mathfrak{h}).\end{aligned} \tag{5.6}$$

The dual generator, acting on the trace class operators, is given by

$$\mathcal{L}_*(\rho)=\sum_{i\in C_m}(L_{i,+}\rho L_{i,+}^*+L_{i,-}\rho L_{i,-}^*)-\rho=\mathcal{M}(\rho)-\rho.$$

A stationary state ρ for the quantum Markov semigroup satisfies $\mathcal{L}_*(\rho)=0$, or $\mathcal{M}(\rho)=\rho$. Therefore, the stationarity for the OQW is the same as that for the quantum Markov semigroup with generator $\mathcal{L}$ in (5.6). The ergodicity for the dynamics $\mathcal{L}$ can be found in some literature, see for instance [11, 12, 17].

Since we are dealing with a finite system ($\mathfrak{h}$ is finite dimensional), a stationary state for $\mathcal{M}$ always exists. Now combining Theorem 3.55, Theorem 3.62, and Remark 3.63 in [17], we can say that

Theorem 5.1 *The stationary states exist uniquely if and only if*

$$\{L_{i,+},\ L_{i,-},\ L_{i,+}^*,\ L_{i,-}^*\ :\ i\in C_m\}'=\mathbb{C}\mathbb{1}. \tag{5.7}$$

Remark 5.2 We remark that the condition (5.7) and the Hypothesis (H) are independent. For example, for the model of B and C in (5.1), the condition (5.7) is satisfied but (H) does not hold. Furthermore, for this model, since $BB^*+CC^*=I_2$, the state $\rho=\frac{1}{2m}\sum_{i\in C_m}I_2\otimes|i\rangle\langle i|$ is the unique stationary state. On the other hand, if we take $B=\frac{1}{\sqrt{2}}\begin{pmatrix}0&1\\1&0\end{pmatrix}$ and $C=\frac{1}{\sqrt{2}}I_2$, then the Hypothesis (H) holds but the equation (5.7) does not hold. And for this model we have two different stationary states: $\rho=\frac{1}{2m}\sum_{i\in C_m}I_2\otimes|i\rangle\langle i|$ and $\eta=\frac{1}{2m}\sum_{i\in C_m}\eta_0\otimes|i\rangle\langle i|$, where $\eta_0=\begin{pmatrix}1&1\\1&1\end{pmatrix}$.

We end this section by giving some examples. In the first example, neither the hypothesis (H) nor the condition (5.7) are fulfilled.

Example 5.3 Let us define a unitary matrix

$$U = \frac{1}{5}\begin{pmatrix} 3 & 4 \\ 4 & -3 \end{pmatrix}.$$

Taking any $0 \le p, q \le 1$ such that $p + q = 1$, define

$$B = \sqrt{p}U, \quad C = \sqrt{q}U.$$

It follows that $B^*B + C^*C = I_2$ and also $BB^* + CC^* = I_2$. Checking, for example, $U^*P_1U = \frac{1}{25}\begin{pmatrix} 9 & 12 \\ 12 & 16 \end{pmatrix}$ we notice that the matrices B and C do not satisfy the hypothesis (H). We first notice that from the property $BB^* + CC^* = I_2$, the state

$$\widetilde{\rho} = \oplus_{i \in C_m} \frac{1}{2m} I_2 \otimes |i\rangle\langle i|$$

is a stationary state for the OQW on C_m generated by B and C. On the other hand, one checks that the matrix $a_0 = \begin{pmatrix} 5 & 2 \\ 2 & 2 \end{pmatrix}$ commutes with U, and so the operator $a = \oplus_{i \in C_m} a_0 \otimes |i\rangle\langle i|$ belongs to the set in the left hand side of (5.7), and hence by Theorem 5.1 the stationary states are not unique. In fact, the state

$$\rho = \oplus_{i \in C_m} \frac{1}{7m} a_0 \otimes |i\rangle\langle i|$$

is also a stationary state different from $\widetilde{\rho}$.

From the above example, one may think that if $\rho = \oplus_{i \in C_m} \rho_i \otimes |i\rangle\langle i|$ is a stationary state, then the ρ_i's are constant: $\rho_i = \rho_j$, $i, j \in C_m$. However, the following example shows that it is not the case, even under the condition (H).

Example 5.4 Consider a cycle C_m of an even number m. Let

$$U = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

With a pair of parameters $0 \le p, q \le 1$ such that $p + q = 1$, define

$$B = \sqrt{p}U, \quad C = \sqrt{q}U.$$

Define a state $\rho = \oplus_{i \in C_m} \rho_i \otimes |i\rangle\langle i|$ by

$$\rho_i = \frac{1}{m}\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \text{ if } i \text{ is even, and } \rho_i = \frac{1}{m}\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \text{ if } i \text{ is odd.}$$

One easily checks that ρ is a stationary state for the OQW on C_m generated by the matrices B and C.

Acknowledgements We are grateful to anonymous referees for their valuable comments, which greatly improved the paper. We thank Mrs. Yoo Jin Cha for drawing the figures. The work of H. J. Yoo was supported by the National Research Foundation of Korea (NRF) Grant funded by the Korean government (MSIT) (No. 2020R1F1A101075).

Author Contributions C.K.K and H.J.Y wrote the main manuscript and all authors reviewd the manuscript.

Declarations

Competing interests The authors declare no competing interests.

Appendix: Generating matrices under Hypothesis (H)

In this appendix, we give the explicit expression of the rank 1 matrices B^*P_iB and C^*P_iC for $i = 1, 2$ under the Hypothesis (H).

Let us denote the unitary matrices U_B and U_C appeared in the singular value decomposition of B and C in (3.5) by

$$U_B = \begin{pmatrix} u_{11} & u_{12} \\ u_{21} & u_{22} \end{pmatrix}, \quad U_C = \begin{pmatrix} v_{11} & v_{12} \\ v_{21} & v_{22} \end{pmatrix}.$$

Lemma A.1 *Let B and C have the singular value decomposition as in* (3.5) *and satisfy the hypothesis (H). Then the rank 1 operators B^*P_iB and C^*P_iC, $i = 1, 2$, have the following forms:*
(i) Case 1: $0 < \lambda < 1$, $0 < \mu < 1$.

$$B^*P_1B = \lambda P_1,\ \ B^*P_2B = \mu P_2 \ \textit{or}\ B^*P_1B = \mu P_2,\ \ B^*P_2B = \lambda P_1,$$

and similarly,

$$\begin{aligned} &C^*P_1C = (1-\lambda)P_1,\ \ C^*P_2C = (1-\mu)P_2 \ \textit{or} \\ &C^*P_1C = (1-\mu)P_2,\ \ C^*P_2C = (1-\lambda)P_1. \end{aligned}$$

(ii) Case 2: $0 < \lambda < 1$, $\mu = 0$.

$$\begin{aligned} &B^*P_1B = \lambda|u_{11}|^2P_1,\ \ B^*P_2B = \lambda|u_{21}|^2P_1,\ \textit{and} \\ &C^*P_1C = P_2,\ \ C^*P_2C = (1-\lambda)P_1 \ \textit{or}\ C^*P_1C = (1-\lambda)P_1,\ \ C^*P_2C = P_2. \end{aligned}$$

(iii) Case 3: $0 < \lambda < 1$, $\mu = 1$.

$$\begin{aligned} &B^*P_1B = P_2,\ \ B^*P_2B = \lambda P_1 \ \textit{or}\ B^*P_1B = \lambda P_1,\ \ B^*P_2B = P_2,\ \textit{and} \\ &C^*P_1C = (1-\lambda)|v_{11}|^2P_1,\ \ C^*P_2C = (1-\lambda)|v_{21}|^2P_1. \end{aligned}$$

(iv) Case 4: $\lambda = 0$, $0 < \mu < 1$.

$$B^*P_1B = \mu|u_{12}|^2P_2,\ \ B^*P_2B = \mu|u_{22}|^2P_2,\ \textit{and}$$

$$C^*P_1C = (1-\mu)P_2,\ C^*P_2C = P_1 \text{ or } C^*P_1C = P_1,\ C^*P_2C = (1-\mu)P_2.$$

(v) Case 5: $\lambda = 1$, $0 < \mu < 1$.

$$B^*P_1B = \mu P_2,\ B^*P_2B = P_1 \text{ or } B^*P_1B = P_1,\ B^*P_2B = \mu P_2,\ \text{and}$$
$$C^*P_1C = (1-\mu)|v_{12}|^2P_2,\ C^*P_2C = (1-\mu)|v_{22}|^2P_2.$$

(vi) Case 6: $\lambda = 0$, $\mu = 1$.

$$B^*P_1B = |u_{12}|^2P_2,\ B^*P_2B = |u_{22}|^2P_2 \text{ and } C^*P_1C = |v_{11}|^2P_1,\ C^*P_2C = |v_{21}|^2P_1.$$

(vii) Case 7: $\lambda = 1$, $\mu = 0$.

$$B^*P_1B = |u_{11}|^2P_1,\ B^*P_2B = |u_{21}|^2P_1 \text{ and } C^*P_1C = |v_{12}|^2P_2,\ C^*P_2C = |v_{22}|^2P_2.$$

(viii) Case 8: $\lambda = 0$, $\mu = 0$ *or* $\lambda = 1$, $\mu = 1$.
These cases result in $B = 0$ *or* $C = 0$*, respectively, and are out of consideration in the model.*

Proof The proof follows easily by using the Hypothesis (H) and the fact that the operators B^*P_iB and C^*P_iC are of rank 1. □

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