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GAUSSIAN QUANTUM MARKOV SEMIGROUPS IN THE FOCK-ANTI-FOCK REPRESENTATION OF WEYL ALGEBRA

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Dedicated to the memory of Professor Habib Ouerdiane

Abstract. We construct Gaussian quantum Markov semigroups on the representation of the Weyl algebra in the tensor product of the Fock space and its conjugate Fock space by the minimal semigroup method. We prove the explicit formula for the action on Weyl operators and study some properties.

1. Introduction

Gaussian Quantum Markov Semigroups (QMSs) are characterized by two equivalent properties: they leave Gaussian states invariant ([16]) and their generator is represented in GKLS form with unbounded operators ([2, 17] and the references therein). Gaussian QMSs arise in several models based on linear couplings of bosonic systems to other bosonic systems with quadratic Hamiltonians and admit several explicit formulae that are useful in many computations exact or approximate. In a series of recent papers ([2, 12] and the references therein) we began a systematic study of their properties such as irreducibility, the structure of invariant states, the structure of the decoherence-free subalgebras and the spectral gap. Moreover, as an application, we showed that a bipartite Gaussian quantum system interacting with an external Gaussian environment may converge towards a unique Gaussian entangled stationary state (see [8]).

All these results were obtained in the Fock representation of the CCR Weyl algebra. In this note we present the construction of Gaussian QMSs in the well-known Fock-anti-Fock or Araki-Woods representation [4]. As we shall see, many formulas and results are expressed in the same way but new features arise because, in the Araki-Woods representation, the CCR algebra has a non-trivial commutant.

The paper is organized as follows. In Section 2 we recall the Fock-anti-Fock representation of the Weyl algebra. Gaussian QMSs on the algebra of all bounded operators on the Hilbert space of the representation are introduced in Section 3. Invariant states and irreducibility are briefly discussed in Sections 5 and 4.

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2. Fock-anti-Fock Representation of Weyl Algebra

In this section we introduce the representation of the Weyl algebra, known as the Araki-Woods representation [4], in the product of boson Fock and its conjugate Fock space. We follow [1] with notations adapted from the reference [15], and we refer to [6] Section 9 for properties. The d-dimensional Hilbert space $\mathbb{C}^d$ is equipped with an inner product $\langle \cdot, \cdot \rangle$, linear in the second component. Denote by $\Gamma_0(\mathbb{C}^d) = \Gamma_0(\mathbb{C}) \otimes \cdots \otimes \Gamma_0(\mathbb{C})$ the Boson Fock space over $\mathbb{C}^d$. Let

$$e_u = \left(1, u, u^{\otimes 2}/\sqrt{2}, u^{\otimes 3}/\sqrt{3}, \cdots\right) \tag{2.1}$$

denote the exponential vector labelled by u. The vector e_0 is called the Fock vacuum. The Fock annihilation and creation operators labelled by $u \in \mathbb{C}^d$ are the mutually adjoint operators in $\Gamma_0(\mathbb{C}^d)$ defined on the dense domain spanned by the exponential vectors by the actions

$$a(u)e_v = \langle u, v \rangle e_v, \quad a^\dagger(u)e_v = \frac{d}{d\epsilon} e_{v+\epsilon u}\Big|_{\epsilon=0}. \tag{2.2}$$

The Fock Weyl operator $W_0(z)$ labelled by $z \in \mathbb{C}^d$ is the unitary operator which acts on the exponential vectors as

$$W_0(z)e_v = e^{-\|z\|^2/2 - \langle z, v \rangle} e_{z+v}, \quad v \in \mathbb{C}^d. \tag{2.3}$$

The map $z \mapsto W_0(z)$ is strongly continuous and forms a representation of the canonical commutation relations (CCR) in that for $z, z' \in \mathbb{C}^d$,

$$W_0(z)W_0(z') = e^{-\mathrm{i}\,\mathrm{Im}\langle z, z' \rangle} W_0(z + z'). \tag{2.4}$$

Given a Hilbert space k, let $\overline{\mathsf{k}}$ denote its dual, or the conjugate Hilbert space of k. For each $f \in \mathsf{k}$ and $T \in \mathcal{B}(\mathsf{k})$, $\overline{f} \in \overline{\mathsf{k}}$ and $\overline{T} \in \mathcal{B}(\overline{\mathsf{k}})$ are defined by

$$\overline{f}(g) = \langle f, g \rangle, \quad \overline{T}\,\overline{f} = \overline{Tf}. \tag{2.5}$$

We make the identification

$$\Gamma_0(\overline{\mathbb{C}^d}) = (\Gamma_0(\mathbb{C}^d))\bar{}$$

in which $e_{\bar{u}} = (e_u)\bar{}$. Let us define the Hilbert space

$$\Gamma(\mathbb{C}^d) := \Gamma_0(\mathbb{C}^d) \otimes (\Gamma_0(\mathbb{C}^d))\bar{}. \tag{2.6}$$

In this space the vacuum vector is defined by

$$E_0 = e_0 \otimes \overline{e}_0. \tag{2.7}$$

Let $Q \in M_d(\mathbb{C})$ be a positive definite $d \times d$ matrix with $Q \geq 1$ (we denote by 1 also the identity matrix). Define

$$Q_+ := \sqrt{(Q+1)/2}, \quad Q_- := \sqrt{(Q-1)/2},$$

so that $Q_+^2 + Q_-^2 = Q$ and $Q_+^2 - Q_-^2 = 1$. For each $z \in \mathbb{C}^d$, define

$$W(z) := W_0(Q_+ z) \otimes \overline{W_0}(-Q_- z). \tag{2.8}$$

It is easily checked that $W(z)$ is a strongly continuous representation of the CCR in $\Gamma(\mathbb{C}^d)$. Denoting $\mathfrak{A}$ the von Neumann algebra generated by the operators $W(u)$, $u \in \mathbb{C}^d$, its commutant $\mathfrak{A}'$ is generated by the operators $W'(z)$, $z \in \mathbb{C}^d$, where

$$W'(z) = W_0(-Q_-z) \otimes \overline{W_0}(Q_+z). \tag{2.9}$$

The vector E_0 is cyclic and separating for both $\mathfrak{A}$ and $\mathfrak{A}'$. Let $\mathfrak{M}$ denote the linear span of the operators $W'(u)$, $u \in \mathbb{C}^d$, which is a *-algebra. The corresponding annihilation and creation operators are respectively defined on $\mathfrak{M}E_0$ by

$$A(u) = a(Q_+u) \otimes I + I \otimes a^\dagger(\overline{Q_-u}), \quad A^\dagger(u) = a^\dagger(Q_+u) \otimes I + I \otimes a(\overline{Q_-u}). \tag{2.10}$$

where I denotes the identity operator on $\Gamma_0(\mathbb{C}^d)$ (see [1, 15] for the details).

Operators $A(u), A^\dagger(u)$ are closable. We will denote their closures by the same symbol.

We also notice that $A(u), A^\dagger(u), W(u)$, as operators on $\mathfrak{M}E_0$, fulfill the following commutation relations for all $u, v \in \mathbb{C}^d$,

$$[A(u),A^\dagger(v)]=\langle u,v\rangle I \otimes I;\ [A(u),W(v)]=\langle u,v\rangle W(v);\ [A^\dagger(u),W(v)]=\langle v,u\rangle W(v) \tag{2.11}$$

3. Gaussian Quantum Markov Semigroups

Let $\mathsf{h} = \Gamma(\mathbb{C}^d)$ be the Hilbert space defined by (2.6). As in [2] we define the pre-generator of Gaussian QMSs on $\mathcal{B}(\mathsf{h})$ in the Gorini-Kossakowski-Lindblad-Sudarshan (GKLS) form as follows:

$$\mathcal{L}(x) = \mathrm{i}[H, x] - \frac{1}{2}\sum_{l=1}^{m}(L_l^* L_l x - 2L_l^* x L_l + x L_l^* L_l), \tag{3.1}$$

where $1 \leq m \leq 2d$, and

$$\begin{aligned} H &= \sum_{j,k=1}^{d}\left(\Omega_{jk}A_j^\dagger A_k + \frac{1}{2}\kappa_{jk}A_j^\dagger A_k^\dagger + \frac{1}{2}\overline{\kappa_{jk}}A_j A_k\right) \\ &\qquad + \sum_{j=1}^{d}\left(\frac{1}{2}\zeta_j A_j^\dagger + \frac{1}{2}\overline{\zeta_j}A_j\right), \end{aligned} \tag{3.2}$$

$$L_l = \sum_{k=1}^{d}\left(\overline{v_{lk}}A_k + u_{lk}A_k^\dagger\right) = A(v_{l\bullet}) + A^\dagger(u_{l\bullet}), \tag{3.3}$$

$\Omega := (\Omega_{jk})_{1\leq j,k\leq d} = \Omega^*$ and $\kappa := (\kappa_{jk})_{1\leq j,k\leq d} = \kappa^T \in M_d(\mathbb{C})$, are $d \times d$ complex matrices with Ω Hermitian and κ symmetric, $V = (v_{lk})_{1\leq l\leq m, 1\leq k\leq d}$, $U = (u_{lk})_{1\leq l\leq m, 1\leq k\leq d} \in M_{m\times d}(\mathbb{C})$ are $m \times d$ matrices and $\zeta = (\zeta_j)_{1\leq j\leq d} \in \mathbb{C}^d$. The notations $v_{l\bullet}$ and $u_{l\bullet}$ denote the vectors in $\mathbb{C}^d$ obtained from the l-th row of the corresponding matrices.

The minimal QMS associated with operators L_l, H can be constructed as in the Fock case. Indeed, one can prove as in [2] Proposition 4.2 that the closure of the operator G defined the algebraic (not closed) linear manifold D generated by finite particle vectors

$$G = -\mathrm{i}H - \frac{1}{2}\sum_{l=1}^{m} L_l^* L_l$$

generates a strongly continuous contraction semigroup $(P_t)_{t\geq 0}$ on h and D is an essential domain for G. Moreover, recalling that the generalized commutant of an unbounded operator L is the set of bounded operators x for which $xL \subset Lx$, and noting that operators H, L_l belong to the generalized commutant of $\mathfrak{A}'$, we deduce that $P_t \in \mathfrak{A}$ for all $t \geq 0$.

The construction of the minimal QMS on $\mathcal{B}(\mathsf{h})$ follows that usual lines. We define recursively linear maps $\mathcal{T}_t^{(n)}$ on $\mathcal{B}(\mathsf{h})$ by

$$\begin{aligned} \mathcal{T}_t^{(0)}(x) &= P_t^* x P_t \\ \langle v, \mathcal{T}_t^{(n+1)}(x)u\rangle &= \langle P_t v, xP_t u\rangle + \sum_{l\geq 1}\int_0^t \left\langle L_l P_{t-s}v, \mathcal{T}_t^{(n)}(x) L_l P_{t-s}u\right\rangle \mathrm{d}s \end{aligned}$$

and, for all $x \geq 0$,

$$\mathcal{T}_t(x) = \sup_{n\geq 0} \mathcal{T}_t^{(n)}(x).$$

Writing $x = (x+x^*)/2 + \mathrm{i}(x-x^*)/(2\mathrm{i})$ and decomposing self-adjoint operators into their positive and negative parts, we can extend the definition of $\mathcal{T}_t$ to arbitrary $x \in \mathcal{B}(\mathsf{h})$. In addition, one can prove that $(\mathcal{T}_t)_{t\geq 0}$ is a semigroup of completely positive maps satisfying

$$\begin{aligned} \langle v, \mathcal{T}_t(x)u\rangle &= \langle v, xu\rangle \\ &+ \int_0^t \Big(\langle Gv, \mathcal{T}_s(x)u\rangle + \sum_{l=1}^m \langle L_l v, \mathcal{T}_s(x) L_l u\rangle + \langle v, \mathcal{T}_s(x)Gu\rangle \Big)\mathrm{d}s \end{aligned}$$

$\forall u, v \in \mathrm{Dom}(G)$, $\forall x \in \mathcal{B}(\mathsf{h})$.

It is worth noticing that, for all $x \in \mathfrak{A}$, $n \geq 0$ and $t \geq 0$, the operator $\mathcal{T}_t^{(n)}(x)$ commutes with all $y \in \mathfrak{A}'$ therefore $\mathcal{T}_t(x) \in \mathfrak{A}$. It follows that the von Neumann algebra $\mathfrak{A}$ is $\mathcal{T}_t$-invariant and one may consider also the QMS obtained by restriction to $\mathfrak{A}$.

Finally, one can show that $\mathcal{T}_t(\mathbb{1}) = \mathbb{1}$ for $t > 0$ applying a standard result on conservativity of minimal QMS as in [2] Section 4.4.

We would like to point out that, in our opinion, the above construction is simpler and more straightforward than the one of the predual QMS on the predual of $\mathfrak{A}$ done in [14] because we do not need to because we are not dealing with the complicated structure of the predual of $\mathfrak{A}$.

We can show as in [2] the explicit formula of the action of the semigroup acting on the Weyl operators. It turns out that the action of the semigroup is exactly the same as that for the Weyl operators of single boson Fock space obtained in [2].

Theorem 3.1. *For all Weyl operator $W(z)$, $z \in \mathbb{C}^d$, we have*

$$\mathcal{T}_t(W(z)) = \exp\left(-\frac{1}{2}\int_0^t \mathrm{Re}\langle \mathrm{e}^{sZ}z, C\mathrm{e}^{sZ}z\rangle ds + \mathrm{i}\int_0^t \mathrm{Re}\langle \zeta, \mathrm{e}^{sZ}z\rangle \mathrm{d}s\right) W(\mathrm{e}^{tZ}z), \tag{3.4}$$

where the real linear operators Z and C on $\mathbb{C}^d$ are

$$Zz = \left((U^T\overline{U} - V^T\overline{V})/2 + i\Omega\right) z + \left((U^T V - V^T U)/2 + i\kappa\right)\overline{z}, \tag{3.5}$$

$$Cz = (U^T\overline{U} + V^T\overline{V})z + (U^T V + V^T U)\overline{z}. \tag{3.6}$$

For the self-containedness, we provide with a proof in the Appendix.

4. Irreducibility

The notion of irreducibility plays a key role in the study of the structure of a QMS. It allows one, for example, when one can find an invariant state, that it is unique. Moreover, it plays a key role in the study of the structure of a QMS (see [7] and the references therein). We begin by recalling the

Definition 4.1. *A projection p is* subharmonic *if $\mathcal{T}_t(p) \geq p$ for all $t \geq 0$. A QMS $\mathcal{T}$ is* irreducible *if its only subharmonic are* 0 *and* $\mathbb{1}$.

Subharmonic projections of a QMS with generator in a GKLS form associated with unbounded operators G, L_l are characterized by the following (see [13] Theorem III.1)

Theorem 4.2. *A projection p is subharmonic for $\mathcal{T}$ if and only if the range $\mathrm{Rg}(p)$ of p is invariant for the operators P_t $(t \geq 0)$ of the strongly continuous contraction semigroup on h generated by G and $L_\ell u = pL_\ell u$, for all $u \in \mathrm{Dom}(G) \cap \mathrm{Rg}(p)$, and all $\ell \geq 1$.*

Leaving aside the technical issues of operator domains it is clear that any projection $p \in \mathfrak{A}'$ commutes with operators L_l and H and so, by (3.1), $\mathcal{L}(p) = 0$. Therefore p is a fixed point for all maps $\mathcal{T}_t$. As a result, the Gaussian QMS $\mathcal{T}$ on $\mathcal{B}(\mathsf{h})$ with H, L_l given by (3.2), (3.3) is *not* irreducible for all choices of $U, V, \Omega, \kappa, \zeta$. On one hand, this constitutes a clear difference with respect to Gaussian QMSs in the Fock representation of the CCR (see [10]), on the other hand, it suggest considering restrictions of completely positive maps $\mathcal{T}_t$ to $\mathfrak{A}$ and modifying Definition 4.1 by adding the condition $p \in \mathfrak{A}$.

5. Invariant States

In this section we point out problems arising in the study of invariant states for the QMS $\mathcal{T}$ we constructed. Starting from the definition of Gaussian states

Definition 5.1. *A normal state ω is called a quantum Gaussian state if there exist $\mu \in \mathbb{C}^d$ and a real linear, symmetric, invertible operator S such that*

$$\widehat{\omega}(z) := \omega(W(z)) = \exp\left(-\frac{1}{2}\mathrm{Re}\langle z, Sz\rangle - \mathrm{i}\,\mathrm{Re}\langle \mu, z\rangle\right) \quad z \in \mathbb{C}^d. \tag{5.1}$$

In that case μ is called the mean vector and S the covariance operator, and Gaussian state ω is also denoted by $\omega_{(\mu,S)}$.

Considering the restriction of the QMS $\mathcal{T}$ to $\mathfrak{A}$, and denoting by $\mathcal{T}_*$ the predual semigroup on $\mathfrak{A}_*$ one can prove as in [16, Theorem 5.1].

Proposition 5.2. *If $\mathcal{T}$ is a Gaussian QMS on $\mathfrak{A}$, then $\mathcal{T}_{*t}(\omega_{(\mu,S)}) = \rho_{(\omega_t,S_t)}$ with*

$$\mu_t = \mathrm{e}^{tZ^\#}\mu - \int_0^t \mathrm{e}^{sZ^\#}\zeta \mathrm{d}s, \quad S_t = \mathrm{e}^{tZ^\#}S\mathrm{e}^{tZ} + \int_0^t \mathrm{e}^{sZ^\#}C\mathrm{e}^{sZ}\mathrm{d}s, \tag{5.2}$$

where $Z^\#$ denotes the adjoint of the real linear operator Z with respect to the scalar product $\mathrm{Re}\langle\cdot,\cdot\rangle$.

Any real linear operator S on $\mathbb{C}^d$ has a form: $Sz = S_1 z + S_2 \overline{z}$ for $z = x + iy \in \mathbb{C}^d$ where S_1 and S_2 are complex linear operators. In this case the operator S can be identified as a real matrix $\mathbf{S}$ acting on $\mathbb{R}^{2d}$ with a relation

$$\mathbf{Sz} = \begin{bmatrix} \mathrm{Re}S_1 + \mathrm{Re}S_2 & \mathrm{Im}S_2 - \mathrm{Im}S_1 \\ \mathrm{Im}S_1 + \mathrm{Im}S_2 & \mathrm{Re}S_1 - \mathrm{Re}S_2 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}, \quad \mathbf{z} = \begin{bmatrix} x \\ y \end{bmatrix}.$$

With the above notation, we expect to get the following results obtained in [11] for Gaussian QMS in the usual Weyl representation of the CCR.

Proposition 5.3. *If the Gaussian QMS $\mathcal{T}$ on $\mathfrak{A}$ has an invariant state, then* $\mathrm{Re}(\lambda) \leq 0$ *for all eigenvalue λ of $\mathbf{Z}$.*

Theorem 5.4. *Suppose that* $\mathrm{sp}(\mathbf{Z}) \cap \mathrm{i}\mathbb{R} = \emptyset$. *If $\mathcal{T}$ has an invariant state, then* $\mathrm{sp}(\mathbf{Z})$ *is contained in the left half-plane, the invariant state is Gaussian with covariance matrix and mean vector*

$$\int_0^\infty \mathrm{e}^{s\mathbf{Z}^*}\mathbf{C}\,\mathrm{e}^{s\mathbf{Z}}\mathrm{d}s, \quad (Z^\#)^{-1}\zeta.$$

Moreover, any initial state converges in trace norm towards this Gaussian invariant state.

6. Conclusion and Outlook

We showed that one can investigate Gaussian QMSs on $\mathcal{B}(\mathsf{h})$, where h is the product of the Fock and anti-Fock Hilbert space of the Araki-Woods representation of the von Neumann algebra $\mathfrak{A}$ of CCR, as in the case where the CCR are represented in the Fock space. The von Neumann algebra $\mathfrak{A}$ turns out to be invariant under completely positive maps $\mathcal{T}_t$, in this way one gets an example of a QMS constructed by the minimal semigroup method that does not act on a von Neumann algebra $\mathcal{B}(\mathsf{h})$ (see also [14]).

One may extend (under some summability conditions) these constructions replacing $\mathbb{C}^d$ by an infinite-dimensional Hilbert space and find a Gaussian QMS on a type III von Neumann algebra. Indeed, it is known ([6] Theorem 40) that $\mathfrak{A}$ is a factor of type III if Q has some continuous spectrum. It would be interesting to study the implications of the emerging non-trivial von Neumann algebraic structure on the fixed-point and decoherence-free subalgebras.

7. Appendix A. Proof of Theorem 3.1

We follow step by step the method given in [2] with a slight modification.

Step 1. Taking a time derivative at $t = 0$ of the formula of (3.4) applied to the vectors $\psi(u) \otimes \psi(\overline{v})$, we compute $\mathcal{L}(W(z))$. First, in the strong topology, we get

$$\left.\frac{d}{dt}W(e^{tZ}z)\right|_{t=0} = W(z)\left(A^\dagger(Zz) - A(Zz) + \mathrm{i}\,\mathrm{Im}\langle z, Zz\rangle\right).$$

We then obtain that $\mathcal{L}(W(z)) = W(z)Y(z)$, where

$$Y(z) = A^\dagger(Zz) - A(Zz) + \mathrm{i}\,\mathrm{Im}\langle z, Zz\rangle - \frac{1}{2}\mathrm{Re}\langle z, Cz\rangle + \mathrm{i}\,\mathrm{Re}\langle \zeta, z\rangle.$$

Step 2. We rewrite the GKLS pre-generator (3.1) as

$$\mathcal{L}(W(z)) = \mathrm{i}[H, W(z)] + \frac{1}{2}\sum_{l=1}^{m}\left(L_l^*[W(z), L_l] + [L_l^*, W(z)]L_l\right).$$

Using the property $\Omega = \Omega^*$, $\kappa = \kappa^T$, and relations (2.11), one gets

$$[W(z), L_l] = -W(z)(\overline{V}z + U\overline{z})_l, \quad [L_l^*, W(z)] = W(z)(V\overline{z} + \overline{U}z)_l. \tag{7.1}$$

And so one gets

$$\begin{aligned}[H, W(z)] &= W(z)\Bigg(A^\dagger(\Omega z + \kappa\overline{z}) + A(\Omega z + \kappa\overline{z}) + \frac{1}{2}\langle z, \Omega z + \kappa\overline{z}\rangle \\ &+ \frac{1}{2}\overline{\langle z, \Omega z + \kappa\overline{z}\rangle} + \mathrm{Re}\langle\zeta, z\rangle\Bigg).\end{aligned}$$

By using (7.1) one finds,

$$\begin{aligned}&\sum_{l=1}^{m}\left(L_l^*[W(z), L_l] + [L_l^*, W(z)]L_l\right) \\ = &-W(z)\sum_{l=1}^{m}\left\{(V\overline{z} + \overline{U}z)_l(\overline{V}z + U\overline{z})_l + L_l^*(\overline{V}z + U\overline{z})_l - L_l(V\overline{z} + \overline{U}z)_l\right\}.\end{aligned}$$

Computing term by term, one gets

$$\begin{aligned}-W(z)\sum_{l=1}^{m}(V\overline{z} + \overline{U}z)_l(\overline{V}z + U\overline{z})_l &= -W(z)\Bigg(\langle z, \overline{V^*V}z + V^TU\overline{z}\rangle \\ &+ \overline{\langle z, \overline{U^*U}z + U^TV\overline{z}\rangle}\Bigg), \\ W(z)\sum_{l=1}^{m}L_l(V\overline{z} + \overline{U}z)_l &= W(z)\Bigg(A^\dagger(\overline{U^*U}z + U^TV\overline{z}) \\ &+ A(\overline{V^*V}z + V^TU\overline{z})\Bigg), \\ -W(z)\sum_{l=1}^{m}L_l^*(\overline{V}z + U\overline{z})_l &= -W(z)\Bigg(A^\dagger(\overline{V^*V}z + V^TU\overline{z}) \\ &+ A(\overline{U^*U}z + U^TV\overline{z})\Bigg).\end{aligned}$$

Gathering all terms one finds that $\mathcal{L}(W(z)) = W(z)X(z)$ for some operator $X(z)$ given by

$$\begin{aligned} X(z) \quad = \quad & A^\dagger \left(\left((\overline{U^*U - V^*V})/2 + \mathrm{i}\Omega \right) z + \left((U^T V - V^T U)/2 + \mathrm{i}\kappa \right) \overline{z} \right) \\ & -A \left(\left((\overline{U^*U - V^*V})/2 + \mathrm{i}\Omega \right) z + \left((U^T V - V^T U)/2 + \mathrm{i}\kappa \right) \overline{z} \right) \\ & +\frac{1}{2}\langle z, \mathrm{i}\Omega z + \mathrm{i}\kappa \overline{z}\rangle - \frac{1}{2}\overline{\langle z, \mathrm{i}\Omega z + \mathrm{i}\kappa\overline{z}\rangle} + \mathrm{i}\,\mathrm{Re}\langle \zeta, z\rangle \\ & -\frac{1}{2}\left(\langle z, \overline{V^*V}z + V^T U \overline{z}\rangle + \overline{\langle z, \overline{U^*U}z + U^T V \overline{z}\rangle} \right). \end{aligned}$$

$X(z)$ and $Y(z)$ coincide for all $z \in \mathbb{C}^d$ and the conclusion of Theorem 3.1 follows.

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